THE POP-SWITCH PLANAR ALGEBRA AND THE JONES-WENZL IDEMPOTENTS

ELLIE GRANO AND STEPHEN BIGELOW

Abstract. The Jones-Wenzl idempotents are elements of the Temperley-Lieb planar algebra that are important, but complicated to write down. We will present a new planar algebra, the pop-switch planar algebra, which contains the Temperley-Lieb planar algebra. It is motivated by Jones' idea of the graph planar algebra of type A_n. In the tensor category of idempotents of the pop-switch planar algebra, the nth Jones-Wenzl idempotent is isomorphic to a direct sum of $n+1$ diagrams consisting of only vertical strands.

1. Introduction

The Temperley-Lieb algebras were first introduced by Temperley and Lieb [LT71] in their work on transfer matrices in statistical mechanics. Vaughn F. R. Jones independently rediscovered Temperley-Lieb algebras in his work on von Neumann algebras [Jon99]. He assembled these algebras together to form the Temperley-Lieb planar algebra, the simplest example of a subfactor planar algebra.

The Jones-Wenzl idempotents, first introduced in [Wen87], are elements of the Temperley-Lieb algebras. One way they arise naturally is in representation theory. The Temperley-Lieb algebras encode the category of representations of $U_q(\mathfrak{sl}_2)$, and the Jones-Wenzl idempotents represent the irreducible representations. Chapters in books have been devoted to them [KL94]. They have been categorified by [CK10] and [FSS10], and generalized [OY97].

While important, the Jones-Wenzl idempotents are difficult to write down explicitly. The nth Jones-Wenzl idempotent is a linear combination of every diagram with n non-intersecting strands. The number of these diagrams is the nth Catalan number. To find the coefficient of a given diagram requires a complicated algorithm originally given by Frankel and Khovanov [FK97] and later written down by Morrison [Mor].

In this paper, we define the pop-switch planar algebra, a new planar algebra that contains the Temperley-Lieb planar algebra. Our original motivation was a diagrammatic treatment of the graph planar algebra introduced by Jones [Jon00]. The pop-switch planar algebra captures with simple diagrams the complicated calculations involved in working with objects in the graph planar algebra.

The main theorem of this paper shows that each Jones-Wenzl idempotent is isomorphic to a direct sum of diagrams with only vertical strands. It is to be hoped that this makes them easier to work with, and gives a new approach to some open problems.

2. Background

For convenience, we work over the field $\mathbb{C}$ and let q be a nonzero complex number that is not a root of unity. Many of the results hold over other fields, but if q is a root of unity the proofs fail due to division by zero.

Definition 2.1. The nth *quantum number* is defined as

$$[n] = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and the *quantum binomial* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]}$$

where $0 \leq k \leq n$ are natural numbers.

We have the following identities.

Lemma 2.2. $[k+l] = [k][l+1] - [k-1][l]$.

Proof. This follows from the definition and a simple computation. □

Corollary 2.3. $\begin{bmatrix} k+l \\ l \end{bmatrix} = [l+1]\begin{bmatrix} k+l-1 \\ l \end{bmatrix} - [k-1]\begin{bmatrix} k+l-1 \\ l-1 \end{bmatrix}$.

Proof. After taking a common denominator and cancelling common terms, this reduces to the previous lemma. □

2.1. Planar algebras.

We won't define planar algebras in great detail. See Jones' original paper [Jon99] for a formal definition. See [MPS08] for a helpful introduction.

We will use what are sometimes called *vanilla* planar algebras. These lack any of the optional extra features or properties that are often included in the definition.

A planar tangle T consists of:

- a disk D called the *output disk*,
- a finite set of disjoint disks D_i called the *input disks* in the interior of D,
- a point called a *basepoint* of ∂D and of each ∂D_i, and
- a collection of disjoint curves called *strands* in D.

The strands can be closed curves, or can have endpoints on ∂D or ∂D_i or both. Apart from the endpoints, the strands lie in the interior of D and do not intersect D_i. The basepoints do not coincide with endpoints of strands. Planar tangles are considered up to isotopy in the plane.

It is sometimes possible to insert a planar tangle T_1 into one of the input disks of another planar tangle T_2 to obtain a new planar tangle. Specifically, this is possible if the number of endpoints on the output disk of T_1 is the same as the number of endpoints on the chosen input disk of T_2. Then we can use an isotopy to make the endpoints match up. This still leaves an ambiguity of how to rotate T_1. The basepoints remove this ambiguity: we require the basepoint of the output disk of T_1 to coincide with the basepoint of the chosen input disk of T_2.

The planar tangles, together with this operation of inserting one planar tangle into an input disk of another, form a rather general type of algebraic gadget called an *operad*. Briefly, a planar algebra is a representation of the operad of planar tangles.

More concretely, a planar algebra $\mathcal{P}$ is a sequence of vector spaces $\mathcal{P}_i$ for $i \geq 0$. Suppose T is a planar tangle with input disks $D_1, \ldots, D_n$. Let d_i be the number of endpoints on ∂D_i and let d be the number of endpoints on ∂D. Suppose $v_i \in \mathcal{P}_{d_i}$ for all i. Then there is an action of T

$$T(v_1, \ldots, v_n) \in \mathcal{P}_d.$$

The action of planar tangles must be multilinear, and it must be compatible with the operad structure in a natural sense.

The definition of a planar algebra may seem complicated. However it formalizes a fairly simple idea, familiar to knot theorists, of tangle-like diagrams that can be glued together in arbitrary planar ways. Perhaps the main novelty is that we allow formal linear combinations of diagrams, which glue together in a multilinear way.

An example might help.

2.2. **The Temperley-Lieb planar algebra.** The simplest planar algebra is the Temperley-Lieb planar algebra $\mathcal{TL}$. The vector space $\mathcal{TL}_i$ is spanned by tangle diagrams that have no input disks and i endpoints on the output disk.

There is one relation. A closed loop in a diagram may be deleted at the expense of multiplying by the scalar $q + q^{-1}$. We call this the *bubble-bursting relation*.

If i is odd then $\mathcal{T}_i$ is zero. A basis for $\mathcal{T}_{2n}$ is given by tangle diagrams that have n strands and no closed loops.

In practice, most planar algebras can be thought of as formal linear combinations of diagrams that are similar to Temperley-Lieb diagrams, but with optional extra features, like crossings, orientations, colors, or vertices.

2.3. **The category corresponding to a planar algebra.** Suppose $\mathcal{P}$ is a planar algebra. We now describe how $\mathcal{P}$ can be thought of as a category. In this context, the input and output disks in the definition of $\mathcal{P}$ should be thought of as rectangles instead of round disks.

The category $\mathbf{C}$ corresponding to $\mathcal{P}$ is as follows.

- The objects are the non-negative integers.
- The morphisms from i to j are the elements of $\mathcal{P}_{i+j}$, thought of as having i endpoints on the bottom of the rectangle and j on the top.
- The composition $f \circ g$ is given by stacking f on top of g.

Let $\mathcal{P}_i^j$ denote $\mathcal{P}_{i+j}$ with the elements treated as morphisms from i to j.

An idempotent is an element p of $\mathcal{P}_n^n$ such that $p^2 = p$.

We can expand the objects in the category by a construction known as the *Karoubi envelope*. This new category $\mathbf{C}'$ is defined as follows.

- The objects of $\mathbf{C}'$ are the idempotents of $\mathbf{C}$.
- The morphisms from p to q are morphisms in $\mathbf{C}$ of the form qxp.

Next, note that $\mathbf{C}$ and $\mathbf{C}'$ are also tensor categories, where $x \otimes y$ is obtained by placing x to the left of y.

Finally, we can define a **matrix category** of $\mathbf{C}'$. The objects are formal direct sums of objects of $\mathbf{C}'$ and the morphisms are formal matrices. Instead of this abstract definition, all we need is the following lemma.

Lemma 2.4. *Suppose p and $q_1, \ldots, q_n$ are idempotents such that*

$$p = q_1 + \cdots + q_n,$$

and $q_i q_j = 0$ *whenever* $i \neq j$. *Then*

$$p \simeq q_1 \oplus \cdots \oplus q_n.$$

2.4. **Jones-Wenzl idempotents.** The Jones-Wenzl idempotent p_n is the unique element of $\mathcal{TL}_n^n$ such that

- $p_n \neq 0$
- $p_n^2 = p_n$
- $ap_n = 0$ if a is any diagram that includes a strand with both endpoints at the bottom of the rectangle.
- $p_n b = 0$ if b is any diagram that includes a strand with both endpoints at the top of the rectangle.

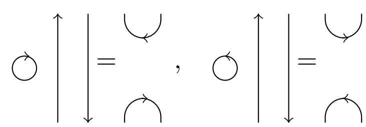
Because of these last two properties, the Jones-Wenzl idempotents are sometimes referred to as "uncappable." If q is a root of unity, the Jones-Wenzl idempotents do not exist for all n.

3. The pop-switch planar algebra

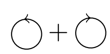
3.1. The pop-switch planar algebra.

Definition 3.1. Let the pop-switch planar algebra $\mathcal{PSPA}$ be the planar algebra generated by oriented strands modulo the following relations.

- The pop-switch relations



- The bubble-bursting relation

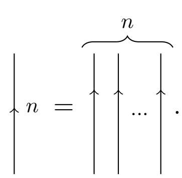
 $= (q + q^{-1})\epsilon$,

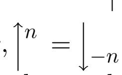
  where ϵ denotes the empty diagram.

This contains the Temperley-Lieb planar algebra; a non-oriented strand is the sum of each orientation.

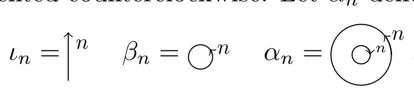
We need some tools to move the diagrams around.

Denote n parallel strands oriented in the same direction by a single oriented strand labelled n.



If n is a negative integer, 

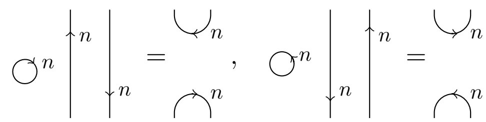
Let ι_n denote n vertical strands oriented up. Let β_n denote n parallel strands that form a bubble oriented counterclockwise. Let α_n denote a β_{-n} inside a β_n.



Lemma 3.2. *Suppose* $x \in \mathcal{PSPA}_0$ *and* y *is a sequence of* $2n$ *vertical strands such that* n *are oriented up and* n *are oriented down. Then* $x \otimes y = y \otimes x$.

Proof. Use the pop-switch relation repeatedly to create a gap and pass x through. Then use the pop-switch relation repeatedly to restore the original $2n$ vertical strands. ☐

Lemma 3.3. The multi-pop-switch relations *The pop-switch relations hold for multiple strands.*



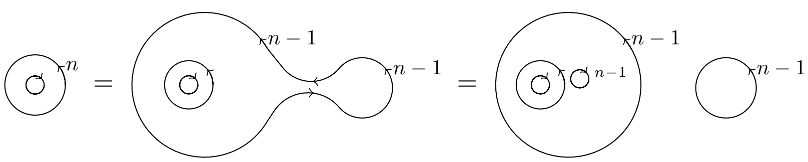
Proof. Without loss of generality, consider the first equality. Induct on n. The case $n = 1$ is the pop-switch relations. For the case $n = k + 1$, move the innermost β_{-k} across two strands using the previous lemma. Then use the case $n = k$, and finally the case $n = 1$. ☐

Corollary 3.4. $\iota_k \otimes \alpha_n = \iota_k$ *and* $\iota_{-k} \otimes \alpha_{-n} = \iota_{-k}$ *for* $k \geq n \geq 0$.

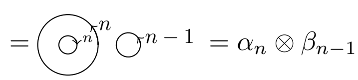
Proof. Consider $\iota_k \otimes \alpha_n$. Use the multi-pop-switch relation by popping the innermost β_n of the α_n. Then straighten out the ι_n. The other case is similar. ☐

Corollary 3.5.

Proof. Start with the left side of the first equality. Use a multi-pop-switch relation on the $n-1$ strands, as shown below.



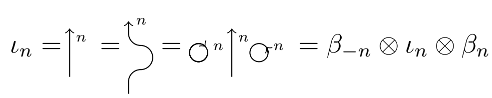
By Lemma 3.2 we can move the β_{-n+1} into the α_1 to achieve the result.



The second identity is proved similarly. ☐

Lemma 3.6. $\iota_n = \beta_{-n} \otimes \iota_n \otimes \beta_n$.

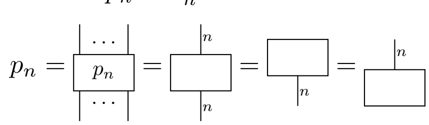
Proof. This follows from the multi-pop switch relations.



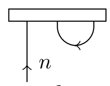
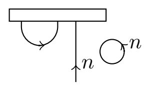
☐

Now we give some relations involving the Jones-Wenzl idempotents p_n. First, we need some notation for them. We will use a rectangle to represent p_n. It should always be assumed that $p_n \in P_n^n$ even if the strands are not drawn.

Notation 3.7.

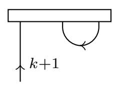
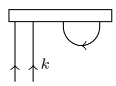
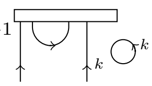


We can make use of the the fact that they are uncappable.

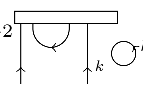
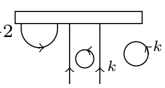
Lemma 3.8.  $= (-1)^{n+1}$  *This relation remains true if all arrows are reversed.*

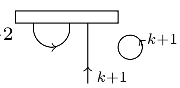
Proof. For the case $n = 0$, use the fact that an unoriented cap gives zero. For the case $n = 1$, use the case $n = 0$ and the pop-switch relation.

For the general case, use induction on n. Start by using the case $n = k$ as follows:

 $=$  $= (-1)^{k+1}$ 

Next use the case $n = 1$, followed by Lemma 3.2, to achieve the result.

$= (-1)^{k+2}$  $= (-1)^{k+2}$ 

$= (-1)^{k+2}$ 

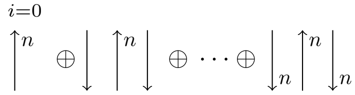
□

4. Proof of the main theorem

The aim of this section is to prove the following.

Theorem 4.1. *The nth Jones-Wenzl idempotent is isomorphic to a direct sum of $n + 1$ diagrams:*

$$p_n \simeq \bigoplus_{i=0}^{n} \iota_{-i} \otimes \iota_n \otimes \iota_{-i}$$

$=$ 

Proof. Since p_n is an idempotent, $p_n = p_n^2 = p_n \mathrm{id}_n p_n$, where id_n is n nonoriented parallel strands, the multiplicative identity in $\mathcal{TL}_n^n$. Now write id_n as a sum of 2^n different ways of orienting n vertical strands. Break this sum into $n+1$ sums depending on how many strands are oriented up.

Definition 4.2. Let p_{n-k}^k denote the sum of $\binom{n}{k}$ diagrams obtained from $p_n \mathrm{id}_n p_n$ by orienting k strands up and $n - k$ strands down in the id_n.

Then $p_n = p_n^0 + p_{n-1}^1 + \cdots + p_0^n$. If $k_1 \neq k_2$, then $p_{n-k_1}^{k_1} p_{n-k_2}^{k_2} = 0$. Thus, by Lemma 2.4,

$$p_n \simeq p_n^0 \oplus p_{n-1}^1 \oplus \cdots \oplus p_0^n.$$

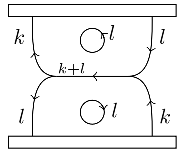
It remains only to show

$$p_l^k \simeq \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}.$$

This is done in Lemma 4.9. □

To prove Lemma 4.9, we first define X_l^k, which we will show is equal a scalar times p_l^k in Lemma 4.8.

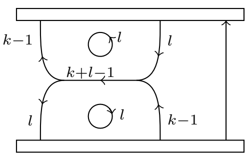
Definition 4.3.

$X_l^k =$ 

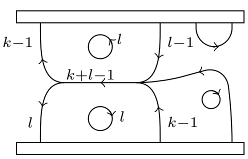
Lemmas 4.4 and 4.5 are similar and begin the inductive step of the proof of Lemma 4.8.

Lemma 4.4. $p_{k+l}(X_l^{k-1} \otimes \iota_1)p_{k+l} = (-1)^l X_l^k \otimes \beta_{-l}$.

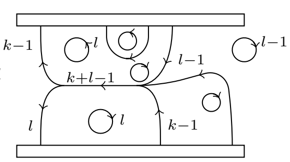
Proof.

$p_{k+l}(X_l^{k-1} \otimes \iota_1)p_{k+l} =$ 

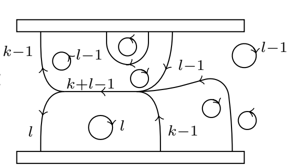
By a pop-switch relation we have the following.

$=$ 

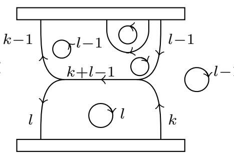
Then by Lemma 3.8 we can move the arc across the $l-1$ strands creating a β_{l-1} on the right. Next we use Lemma 3.6 to replace the arc with $\beta_{-1} \otimes \iota_1 \otimes \beta_1$.

$= (-1)^l$ 

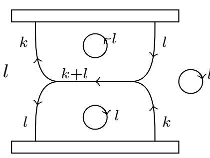
Move the innermost β_1 from the β_l to the far right across $l-1$ strands in both directions by Lemma 3.2.

$= (-1)^l$ 

Then remove the two bubbles on the bottom right of the diagram by Lemma 3.6.

$= (-1)^l$ 

Lastly, by Lemma 3.2 move the β_{l-1} into the β_1 and the β_{-1} into the $\beta_{-(l-1)}$.

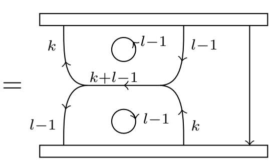
$= (-1)^l$ 

$$= (-1)^l X_l^k \otimes \beta_{-l}$$

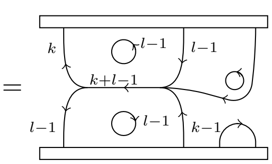
□

Lemma 4.5. $p_{k+l}(X_{l-1}^k \otimes \iota_{-1})p_{k+l} = (-1)^k X_l^k \otimes \beta_k$.

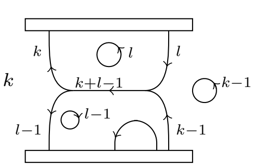
Proof.

$p_{k+l}(X_{l-1}^k \otimes \iota_{-1})p_{k+l} =$ 

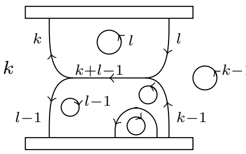
By a pop-switch relation we have the following.

$=$ 

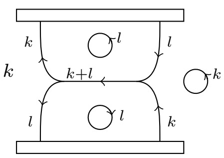
Then by Lemma 3.8 we can move the arc across the $k-1$ strands creating a $\beta_{-(k-1)}$ on the right. Next we use Lemma 3.2 to move the β_1 across the $l-1$ strands in both directions into the β_{l-1}.

$= (-1)^k$ 

Replace the arc with $\beta_{-1} \otimes \iota_1 \otimes \beta_1$.

$$= (-1)^k$$ 

Lastly, by Lemma 3.2 move the β_1 across the $k-1$ strands in both directions into the β_{k-1}. By the same lemma, move the $\beta_{-(l-1)}$ into the β_{-1}.

$$= (-1)^k$$ 

$$= (-1)^k X_l^k \otimes \beta_k$$

□

Lemma 4.6 is the key to proving Lemma 4.7, which is required to complete the proof of Lemma 4.8. It is worth noting that in Lemma 4.7 the X_l^k merely acts as a catalyst to provide enough strands to use 4.6. All that is necessary is the presence of ι_{-l+1} and ι_{k-1} on the left as specified in Lemma 4.6 for the purpose of implementing Corollary 3.4.

Lemma 4.6. *For $k \geq n-1$,*

$$\iota_k \otimes \beta_n = [n]\iota_k \otimes \beta_1 - [n-1]\iota_k$$

and

$$\iota_{-k} \otimes \beta_{-n} = [n]\iota_{-k} \otimes \beta_{-1} - [n-1]\iota_{-k}$$

Proof. We prove the first identity, since the second is similar. Consider the case $n = 2$ with $k \geq 1$. Use the bubble-bursting relation on the innermost loop of β_2. Corollary 3.4 then gives the result.

$$\iota_k \otimes \beta_2 = [2]\iota_k \otimes \beta_1 - \iota_k \otimes \alpha_1 = [2]\iota_k \otimes \beta_1 - \iota_k$$

Now assume $k \geq n-1$. Use the bubble-bursting relation on the innermost loop of β_n. Corollary 3.5, Corollary 3.4, and induction give

$$\begin{aligned}
\iota_k \otimes \beta_n &= [2]\iota_k \otimes \beta_{n-1} - \iota_k \otimes \beta_{n-2} \\
&= [2]([n-1]\iota_k \otimes \beta_1 - [n-2]\iota_k) - ([n-2]\iota_k \otimes \beta_1 - [n-3]\iota_k) \\
&= ([2][n-1] - [n-2])\iota_k \otimes \beta_1 - ([2][n-] - [n-3])\iota_k \\
&= [n]\iota_k \otimes \beta_1 - [n-1]\iota_k
\end{aligned}$$

□

Lemma 4.7. *If $k+l=n$ then*
$$\begin{bmatrix} n-1 \\ l \end{bmatrix} X_l^k \otimes \beta_{-l} + \begin{bmatrix} n-1 \\ k \end{bmatrix} X_l^k \otimes \beta_k = \begin{bmatrix} n \\ k \end{bmatrix} X_l^k.$$

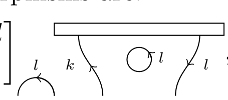
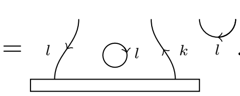
Proof. Note that every term in the equation contains X_l^k. However, the result will hold so long as there are both a ι_{-l+1} and ι_{k-1} on the left of each diagram in order to use Lemma 4.6. Thus it suffices to prove
$$\begin{bmatrix} n-1 \\ l \end{bmatrix} ([l]\beta_{-1} - [l-1]) + \begin{bmatrix} n-1 \\ k \end{bmatrix} ([k]\beta_1 - [k-1]) = \begin{bmatrix} n \\ k \end{bmatrix}.$$
Use the identity
$$\begin{bmatrix} n-1 \\ l \end{bmatrix} [l] = \begin{bmatrix} n-1 \\ k \end{bmatrix} [k],$$
and the bubble bursting relation $\beta_{-1} + \beta_1 = [2]$ to eliminate β_{-1} and β_1 from the left side. Then simplify further using the identity $[2][l] - [l-1] = [l+1]$. We obtain
$$\begin{bmatrix} n-1 \\ l \end{bmatrix} [l+1] - \begin{bmatrix} n-1 \\ k \end{bmatrix} [k-1].$$
By Corollary 2.3, this is equal to $\begin{bmatrix} n \\ k \end{bmatrix}$, as desired. ∎

Lemma 4.8. $p_l^k = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} X_l^k$

Proof. Induct on $n = k+l$. Notice $p_0^1 = \iota_1 = X_0^1$ and $p_1^0 = \iota_{-1} = X_1^0$. Assume $k > 0$ and $l > 0$. Then
$$p_l^k = p_{k+l}(p_l^{k-1} \otimes \iota_1) p_{k+l} + p_{k+l}(p_{l-1}^k \otimes \iota_{-1}) p_{k+l}$$
By Lemma 4.4 and Lemma 4.5,
$$= (-1)^{kl} \begin{bmatrix} k+l-1 \\ l \end{bmatrix} X_l^k \otimes \beta_{-l} + (-1)^{kl} \begin{bmatrix} k+l-1 \\ k \end{bmatrix} X_l^k \otimes \beta_k$$
By Lemma 4.7,
$$= (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} X_l^k$$
∎

Lemma 4.9. $p_l^k \simeq \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$.

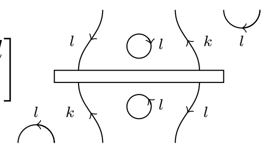
Proof. The explicit isomorphisms are:

$f = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix}$  , $\qquad g =$  .

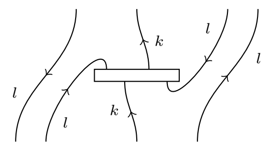
Then $f \circ g = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} X_l^k = p_l^k$ by Lemma 4.8. Thus $f \circ g$ is the identity morphism from p_l^k to p_l^k.

On the other hand, $g \circ f = \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$, the identity morphism from $\iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$ to $\iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$.

$$g \circ f = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix}$$

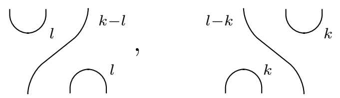


$$= (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix}$$



$$= \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$$

The second equality holds by performing two multi-pop-switch relations: one on the β_{-l} at the top with the l strands to the left and the l strands on the bottom left, and the other on the β_l and the l strands on the right and top right. Now expand the Jones-Wenzl idempotent. The only non-zero term come from one of the following Temperley-Lieb diagrams.



Thus the result of $g \circ f$ must be a scalar times $\iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$. Since $f \circ g$ is the identity and $g \circ f$ is a scalar times the identity, that scalar must be 1. □

5. Graph planar algebra and the Temperley-Lieb planar algebra

This section is motivation for the definition of the pop-switch planar algebra. We start with a summary of the definition of the graph planar algebra, first defined in [Jon00].

Throughout this section, fix a simple graph Γ. For Jones, all planar algebras are shaded, and Γ is required to be bipartite. We will ignore this issue.

Let μ be a function from the vertices of Γ to $\mathbf{R}_{>0}$. We will define the graph planar algebra $\mathcal{P}$ corresponding to (Γ, μ).

For each $k > 0$, let $\mathcal{P}_{2k}$ be the vector space of complex valued functions on the set of loops of length $2k$ on Γ.

Suppose T is a tangle. For each input disk of T, let v_b be a corresponding input vector. We must define a corresponding output vector v. Thus we must define $v(\gamma)$ for every loop γ in Γ that has length equal to the number of endpoints on the outer boundary of T.

A *state* σ of T is a function from the set of regions of T to the set of vertices of Γ such that adjacent regions are sent to adjacent vertices.

Suppose r is a region of T. This is a planar surface with boundary that may include some right-angled corners. The *Euler measure* $e(r)$ is defined in a similar way to the Euler characteristic, using the usual formula $V - E + F$ for a triangulation of r. The difference is, every corner must be a vertex and only counts as $\frac{1}{4}$, any other vertex on a boundary only counts as $\frac{1}{2}$, and every edge on a boundary only counts as $\frac{1}{2}$.

We are finally ready to define the image vector v of the vectors v_b under the action of the tangle T.

$$v(\gamma) = \sum_{\sigma} \left(\prod_{r} \mu(\sigma(r))^{e(r)} \right) \left(\prod_{b} v_b(\sigma|_{\partial b}) \right).$$

The sum is over all states σ that are compatible with γ. The first product is over all regions r of T. The second product is over all input disks b of T.

The Temperley-Lieb planar algebra is a subfactor planar algebra of type A_∞. It can be found inside the graph planar algebra associated to $\Gamma = A_\infty$, which is the ray with vertices indexed by positive integers. The function μ assigns the quantum integer $[n]$ to the nth vertex. (Note we are still assuming q is not a root of unity. If q is a primitive $(n+1)$th root of unity then we should use the graph A_n.)

Suppose T is an oriented tangle. Define a state of T to be a function from the set of regions of T to the set of vertices of A_∞ such that, for any strand of T, if the region to its right is sent to vertex n then the region to its left is sent to vertex $n+1$. Thus, a state is determined by the vertex associated to a single region. In a sense, the orientation on the strands removes the ambiguity in the state of a Temperley-Lieb diagram.

Now suppose T and T' differ by a pop-switch relation. There is an obvious correspondence between states of T and states off T'. Furthermore, the total Euler measure of the region associated to any given vertex is the same. We therefore have a well-defined embedding of the pop-switch planar algebra in the graph planar algebra of the graph A_∞.

One can think of the pop-switch planar algebra as a diagrammatic way to keep track of computations inside the graph planar algebra of A_∞.

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